

# Additional **multcomp** Examples

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It is assumed that the reader is familiar with the theory and applications described in the `generalsiminf` vignette.

## 1 Simple Examples

**Example: Simple Linear Model.** Consider a simple univariate linear model regressing the distance to stop on speed for 50 cars:

```
R> lm.cars <- lm(dist ~ speed, data = cars)
R> summary(lm.cars)
```

Call:

```
lm(formula = dist ~ speed, data = cars)
```

Residuals:

```
    Min      1Q  Median      3Q     Max
-29.07  -9.53  -2.27   9.21  43.20
```

Coefficients:

```
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  -17.579      6.758   -2.60  0.012 *
speed           3.932      0.416    9.46 1.5e-12 ***
```

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 15.4 on 48 degrees of freedom

Multiple R-squared: 0.651, Adjusted R-squared: 0.644

F-statistic: 89.6 on 1 and 48 DF, p-value: 1.49e-12

The estimates of the regression coefficients  $\beta$  and their covariance matrix can be extracted from the fitted model via:

```
R> betahat <- coef(lm.cars)
R> Vbetahat <- vcov(lm.cars)
```

At first, we are interested in the hypothesis  $\beta_1 = 0$  and  $\beta_2 = 0$ . This is equivalent to the linear hypothesis  $\mathbf{K}\beta = 0$  where  $\mathbf{K} = \text{diag}(2)$ , i.e.,

```
R> K <- diag(2)
R> Sigma <- diag(1 / sqrt(diag(K %*% Vbetahat %*% t(K))))
R> z <- Sigma %*% K %*% betahat
R> Cor <- Sigma %*% (K %*% Vbetahat %*% t(K)) %*% t(Sigma)
```

Note that  $\mathbf{z} = (-2.6011, 9.464)$  is equal to the  $t$  statistics. The multiplicity-adjusted  $p$  values can now be computed by means of the multivariate  $t$  distribution utilizing the `pmvt` function available in package `mvtnorm`:

```
R> library("mvtnorm")
R> df.cars <- nrow(cars) - length(betahat)
R> sapply(abs(z), function(x) 1 - pmvt(-rep(x, 2), rep(x, 2), corr = Cor, df = df.cars))

[1] 1.661e-02 2.457e-12
```

Note that the  $p$  value of the global test is the minimum  $p$  value of the partial tests.

The computations above can be performed much more conveniently using the functionality implemented in package `multcomp`. The function `glht` just takes a fitted model and a matrix defining the linear functions, and thus hypotheses, to be tested:

```
R> library("multcomp")
R> cars.ht <- glht(lm.cars, linfct = K)
R> summary(cars.ht)
```

#### Simultaneous Tests for General Linear Hypotheses

```
Fit: lm(formula = dist ~ speed, data = cars)
```

Linear Hypotheses:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept) == 0	-17.579	6.758	-2.60	0.017 *
speed == 0	3.932	0.416	9.46	<1e-10 ***

```
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported -- single-step method)
```

Simultaneous confidence intervals corresponding to this multiple testing procedure are available via

```
R> confint(cars.ht)
```

#### Simultaneous Confidence Intervals

```
Fit: lm(formula = dist ~ speed, data = cars)
```

```
Quantile = 2.131
```

95% family-wise confidence level

Linear Hypotheses:

	Estimate	lwr	upr
(Intercept) == 0	-17.579	-31.981	-3.177
speed == 0	3.932	3.047	4.818

The application of the framework isn't limited to linear models, nonlinear least-squares estimates can be tested as well. Consider constructing simultaneous confidence intervals for the model parameters (example from the manual page of `nls`):

```
R> DNase1 <- subset(DNase, Run == 1)
R> fm1DNase1 <- nls(density ~ SSlogis(log(conc), Asym, xmid, scal), DNase1)
R> K <- diag(3)
R> rownames(K) <- names(coef(fm1DNase1))
R> confint(glht(fm1DNase1, linfct = K))
```

Simultaneous Confidence Intervals

```
Fit: nls(formula = density ~ SSlogis(log(conc), Asym, xmid, scal),
  data = DNase1, algorithm = "default", control = list(maxiter = 50,
  tol = 1e-05, minFactor = 0.0009765625, printEval = FALSE,
  warnOnly = FALSE, scaleOffset = 0, nDcentral = FALSE),
  trace = FALSE)
```

Quantile = 2.136

95% family-wise confidence level

Linear Hypotheses:

	Estimate	lwr	upr
Asym == 0	2.345	2.178	2.512
xmid == 0	1.483	1.309	1.657
scal == 0	1.041	0.973	1.110

which is not totally different from univariate confidence intervals

```
R> confint(fm1DNase1)
```

	2.5%	97.5%
Asym	2.1935	2.539
xmid	1.3215	1.679
scal	0.9743	1.115

because the parameter estimates are highly correlated

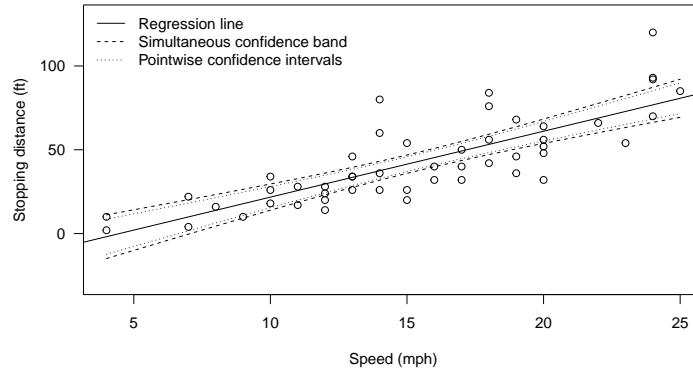


Figure 1: `cars` data: Regression line with confidence bands (dashed) and intervals (dotted).

```
R> cov2cor(vcov(fm1DNase1))
```

```
      Asym  xmid  scal
Asym 1.0000 0.9868 0.9008
xmid 0.9868 1.0000 0.9063
scal 0.9008 0.9063 1.0000
```

**Example: Confidence Bands for Regression Line.** Suppose we want to plot the linear model fit to the `cars` data including an assessment of the variability of the model fit. This can be based on simultaneous confidence intervals for the regression line  $x_i^T \hat{\beta}$ :

```
R> K <- model.matrix(lm.cars)[!duplicated(cars$speed),]
R> ci.cars <- confint(glm(lm.cars, linfct = K), abseps = 0.1)
```

Figure 1 depicts the regression fit together with the confidence band for the regression line and the pointwise confidence intervals as computed by `predict(lm.cars)`.

## 2 Multiple Comparison Procedures

Multiple comparisons of means, i.e., regression coefficients for groups in AN(C)OVA models, are a special case of the general framework sketched in the previous section. The main difficulty is that the comparisons one is usually interested in, for example all-pairwise differences, can't be directly specified based on model

parameters of an AN(C)OVA regression model. We start with a simple one-way ANOVA example and generalize to ANCOVA models in the following.

Consider a one-way ANOVA model, i.e., the only covariate  $x$  is a factor at  $j$  levels. In the absence of an intercept term only, the elements of the parameter vector  $\beta \in \mathbb{R}^j$  correspond to the mean of the response in each of the  $j$  groups:

```
R> ex <- data.frame(y = rnorm(12), x = gl(3, 4, labels = LETTERS[1:3]))
R> aov.ex <- aov(y ~ x - 1, data = ex)
R> coef(aov.ex)
```

```
      xA      xB      xC
0.1627 0.1835 0.2976
```

Thus, the hypotheses  $\beta_2 - \beta_1 = 0$  and  $\beta_3 - \beta_1 = 0$  can be written in form of a linear function  $\mathbf{K}\beta$  with

```
R> K <- rbind(c(-1, 1, 0),
+            c(-1, 0, 1))
R> rownames(K) <- c("B - A", "C - A")
R> colnames(K) <- names(coef(aov.ex))
R> K
```

```
      xA xB xC
B - A -1  1  0
C - A -1  0  1
```

Using the general linear hypothesis function `glht`, this so-called ‘many-to-one comparison procedure’ [Dunnett, 1955] can be performed via

```
R> summary(glht(aov.ex, linfct = K))
```

#### Simultaneous Tests for General Linear Hypotheses

```
Fit: aov(formula = y ~ x - 1, data = ex)
```

Linear Hypotheses:

	Estimate	Std. Error	t value	Pr(> t )
B - A == 0	0.0208	0.8010	0.03	1.00
C - A == 0	0.1349	0.8010	0.17	0.98

(Adjusted p values reported -- single-step method)

Alternatively, a symbolic description of the general linear hypothesis of interest can be supplied to `glht`:

```
R> summary(glht(aov.ex, linfct = c("xB - xA = 0", "xC - xA = 0")))
```

#### Simultaneous Tests for General Linear Hypotheses

```
Fit: aov(formula = y ~ x - 1, data = ex)
```

Linear Hypotheses:

```
              Estimate Std. Error t value Pr(>|t|)
xB - xA == 0    0.0208     0.8010    0.03    1.00
xC - xA == 0    0.1349     0.8010    0.17    0.98
(Adjusted p values reported -- single-step method)
```

However, in the presence of an intercept term, the full parameter vector  $\theta = c(\mu, \beta_1, \dots, \beta_j)$  can't be estimated due to singularities in the corresponding design matrix. Therefore, a vector of *contrasts*  $\gamma^*$  of the original parameter vector  $\gamma$  is fitted. More technically, a contrast matrix  $\mathbf{C}$  is included into this model such that  $\beta = \mathbf{C}\gamma^*$  any we only obtain estimates for  $\gamma^*$ , but not for  $\gamma$ :

```
R> aov.ex2 <- aov(y ~ x, data = ex)
R> coef(aov.ex2)
```

```
(Intercept)          xB          xC
      0.16273      0.02081      0.13488
```

The default contrasts in R are so-called treatment contrasts, nothing but differences in means for one baseline group (compare the Dunnett contrasts and the estimated regression coefficients):

```
R> contr.treatment(table(ex$x))
```

```
 4 4
4 0 0
4 1 0
4 0 1
```

```
R> K %*% contr.treatment(table(ex$x)) %*% coef(aov.ex2)[-1]
```

```
      [,1]
B - A 0.02081
C - A 0.13488
```

so that  $\mathbf{KC}\hat{\beta}^* = \mathbf{K}\hat{\beta}$ .

When the `mcp` function is used to specify linear hypotheses, the `glht` function takes care of contrasts. Within `mcp`, the matrix of linear hypotheses  $\mathbf{K}$  can be written in terms of  $\gamma$ , not  $\gamma^*$ . Note that the matrix of linear hypotheses only applies to those elements of  $\hat{\gamma}^*$  attached to factor `x` but not to the intercept term:

```
R> summary(glht(aov.ex2, linfct = mcp(x = K)))
```

Simultaneous Tests for General Linear Hypotheses

Multiple Comparisons of Means: User-defined Contrasts

```
Fit: aov(formula = y ~ x, data = ex)
```

```
Linear Hypotheses:
```

	Estimate	Std. Error	t value	Pr(> t )
B - A == 0	0.0208	0.8010	0.03	1.00
C - A == 0	0.1349	0.8010	0.17	0.98

(Adjusted p values reported -- single-step method)

or, a little bit more convenient in this simple case:

```
R> summary(glht(aov.ex2, linfct = mcp(x = c("B - A = 0", "C - A = 0"))))
```

Simultaneous Tests for General Linear Hypotheses

Multiple Comparisons of Means: User-defined Contrasts

```
Fit: aov(formula = y ~ x, data = ex)
```

```
Linear Hypotheses:
```

	Estimate	Std. Error	t value	Pr(> t )
B - A == 0	0.0208	0.8010	0.03	1.00
C - A == 0	0.1349	0.8010	0.17	0.98

(Adjusted p values reported -- single-step method)

More generally, inference on linear functions of parameters which can be interpreted as ‘means’ are known as *multiple comparison procedures* (MCP). For some of the more prominent special cases, the corresponding linear functions can be computed by convenience functions part of **multcomp**. For example, Tukey all-pair comparisons for the factor *x* can be set up using

```
R> glht(aov.ex2, linfct = mcp(x = "Tukey"))
```

General Linear Hypotheses

Multiple Comparisons of Means: Tukey Contrasts

```
Linear Hypotheses:
```

	Estimate
B - A == 0	0.0208
C - A == 0	0.1349
C - B == 0	0.1141

The initial parameterization of the model is automatically taken into account:

```
R> glht(aov.ex, linfct = mcp(x = "Tukey"))
```

## General Linear Hypotheses

Multiple Comparisons of Means: Tukey Contrasts

Linear Hypotheses:

	Estimate
B - A == 0	0.0208
C - A == 0	0.1349
C - B == 0	0.1141

### 3 Two-way ANOVA

For two-way ANOVA models, one might be interested in comparing the levels of the two factors simultaneously. For the model

```
R> mod <- lm(breaks ~ wool + tension, data = warpbreaks)
```

one can extract the appropriate contrast matrices for both factors via

```
R> K1 <- glht(mod, mcp(wool = "Tukey"))$linfct
R> K2 <- glht(mod, mcp(tension = "Tukey"))$linfct
```

and we can simultaneously compare the levels of each factor using

```
R> summary(glht(mod, linfct = rbind(K1, K2)))
```

#### Simultaneous Tests for General Linear Hypotheses

Fit: `lm(formula = breaks ~ wool + tension, data = warpbreaks)`

Linear Hypotheses:

	Estimate	Std. Error	t value	Pr(> t )
B - A == 0	-5.78	3.16	-1.83	0.2304
M - L == 0	-10.00	3.87	-2.58	0.0458 *
H - L == 0	-14.72	3.87	-3.80	0.0015 **
H - M == 0	-4.72	3.87	-1.22	0.5703

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1  
(Adjusted p values reported -- single-step method)

where the first comparison is with respect to `wool` and the remaining three to `tension`.

For models with interaction term

```
R> mod <- lm(breaks ~ wool * tension, data = warpbreaks)
```

one might be interested in comparing the levels of `tension` within the levels of `wool`. There are two ways to do this. First, we compute the means of the response for all six combinations of the levels of `wool` and `tension`:



```
R> tmp <- expand.grid(tension = unique(warbreaks$tension),
+                   wool = unique(warbreaks$wool))
R> X <- model.matrix(~ wool * tension, data = tmp)
R> glht(mod, linfct = X)
```

#### General Linear Hypotheses

Linear Hypotheses:

	Estimate
1 == 0	44.6
2 == 0	24.0
3 == 0	24.6
4 == 0	28.2
5 == 0	28.8
6 == 0	18.8

which is the same as

```
R> predict(mod, newdata = tmp)
```

	1	2	3	4	5	6
	44.56	24.00	24.56	28.22	28.78	18.78

We now construct a contrast matrix based on Tukey-contrasts for `tension` in a block-diagonal way, i.e., for each level of `wool`:

```
R> Tukey <- contrMat(table(warbreaks$tension), "Tukey")
R> K1 <- cbind(Tukey, matrix(0, nrow = nrow(Tukey), ncol = ncol(Tukey)))
R> rownames(K1) <- paste(levels(warbreaks$wool)[1], rownames(K1), sep = ":")
R> K2 <- cbind(matrix(0, nrow = nrow(Tukey), ncol = ncol(Tukey)), Tukey)
R> rownames(K2) <- paste(levels(warbreaks$wool)[2], rownames(K2), sep = ":")
R> K <- rbind(K1, K2)
R> colnames(K) <- c(colnames(Tukey), colnames(Tukey))
```

and perform the tests via

```
R> summary(glht(mod, linfct = K %*% X))
```

#### Simultaneous Tests for General Linear Hypotheses

```
Fit: lm(formula = breaks ~ wool * tension, data = warbreaks)
```

Linear Hypotheses:

	Estimate	Std. Error	t value	Pr(> t )
A:M - L == 0	-20.556	5.157	-3.99	0.0013 **
A:H - L == 0	-20.000	5.157	-3.88	0.0018 **
A:H - M == 0	0.556	5.157	0.11	1.0000
B:M - L == 0	0.556	5.157	0.11	1.0000

```

B:H - L == 0   -9.444      5.157   -1.83   0.3080
B:H - M == 0  -10.000     5.157   -1.94   0.2553
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported -- single-step method)

```

where the effects are the same as

```
R> K %%% predict(mod, newdata = tmp)
```

```

      [,1]
A:M - L -20.5556
A:H - L -20.0000
A:H - M  0.5556
B:M - L  0.5556
B:H - L -9.4444
B:H - M -10.0000

```

We see that the groups (M, L) and (H, L) are different, however, only for wool A (in contrast to the additive model above).

Note that the same results can be obtained by fitting the so-called cell-means model based on a new factor derived as the interaction of `wool` and `tension`:

```

R> warpbreaks$tw <- with(warpbreaks, interaction(tension, wool))
R> cell <- lm(breaks ~ tw - 1, data = warpbreaks)
R> summary(glht(cell, linfct = K))

```

#### Simultaneous Tests for General Linear Hypotheses

```
Fit: lm(formula = breaks ~ tw - 1, data = warpbreaks)
```

Linear Hypotheses:

	Estimate	Std. Error	t value	Pr(> t )	
A:M - L == 0	-20.556	5.157	-3.99	0.0013	**
A:H - L == 0	-20.000	5.157	-3.88	0.0018	**
A:H - M == 0	0.556	5.157	0.11	1.0000	
B:M - L == 0	0.556	5.157	0.11	1.0000	
B:H - L == 0	-9.444	5.157	-1.83	0.3080	
B:H - M == 0	-10.000	5.157	-1.94	0.2553	

```

---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported -- single-step method)

```

## 4 Test Procedures

Several global and multiple test procedures are available from the `summary` method of `glht` objects and can be specified via its `test` argument:

- `test = univariate()` univariate  $p$  values based on either the  $t$  or normal distribution are reported. Controls the type I error for each partial hypothesis only.
- `test = Ftest()` global  $F$  test for  $H_0$ .
- `test = Chisqtest()` global  $\chi^2$  test for  $H_0$ .
- `test = adjusted()` multiple test procedures as specified by the `type` argument to `adjusted`: "`single-step`" denotes adjusted  $p$  values as computed from the joint normal or  $t$  distribution of the  $\mathbf{z}$  statistics (default), "`free`" implements multiple testing procedures under free combinations, "`Shaffer`" implements Bonferroni-adjustments taking logical constraints into account Shaffer [1986] and "`Westfall`" takes both logical constraints and correlations among the  $\mathbf{z}$  statistics into account Westfall [1997]. In addition, all adjustment methods implemented in `p.adjust` can be specified as well.

## 5 Quality Assurance

The analyses shown in Westfall et al. [1999] can be reproduced using `multcomp` by running the R transcript file in `inst/MCMT`.

## References

- Charles W. Dunnett. A multiple comparison procedure for comparing several treatments with a control. *Journal of the American Statistical Association*, 50(272):1096–1121, 1955.
- Juliet P. Shaffer. Modified sequentially rejective multiple test procedures. *Journal of the American Statistical Association*, 81:826–831, 1986.
- Peter H. Westfall. Multiple testing of general contrasts using logical constraints and correlations. *Journal of the American Statistical Association*, 92:299–306, 1997.
- Peter H. Westfall, Randall D. Tobias, Dror Rom, Russell D. Wolfinger, and Yosef Hochberg. *Multiple Comparisons and Multiple Tests Using the SAS System*. SAS Institute, Cary, NC, 1999.